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ON PROPER TRANSONIC FLOWS OF RADIATIVE GAS IN CHANNELS WITH SLIGHTLY VARYING PARAMETERS

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Flows of gas at velocities close to the isentropic and isothermal speeds of sound in channels with slowly changing temperature and curved walls are considered. The model takes into account the convection nonlinearity resulting from the cumulative effect of perturbation propagation. It also permits the analysis of the arbitrary effect of radiation on the motion of gas. The derived nonlinear system of equations defines the flow of gas in channels whose transverse optical thickness is of the order of unity. Similar equations for quasi-isentropic flows appear in [1].

1. Input equations. Let us consider the stationary equilibrium flow of an inviscid non-heat-conducting radiative gas in a channel with plane or axial symmetry. The channel walls are assumed to be nearly parallel planes or, in the case of axial symmtry, to have a nearly cylindrical surface. In the plane case the channels are assumed to be symmetric about the plane $y = 0$.

We assume that the input of radiation to the internal energy density and to the pressure is small. The motion of such medium is defined by the equations

$$\frac{\partial \rho u}{\partial x} + \frac{1}{y^{d-1}} \frac{\partial}{\partial y} y^{d-1} \rho v = 0 \tag{1.1}$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0, \quad u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{1}{\rho} \frac{\partial p}{\partial y} = 0$$

$$\rho T \left(u \frac{\partial s}{\partial x} + v \frac{\partial s}{\partial y} \right) + \operatorname{div} \mathbf{q} = 0$$

where p is the pressure, ρ is the density, T is the temperature, s is the entropy of gas, $\mathbf{q}$ is the flux of radiant energy, and u and v are the gas velocity components along the lengthwise and transverse coordinates x and y, respectively. In the plane and axisymmetric cases $d = 1$ and $d = 2$, respectively.

The radiant energy flux is defined by

$$\mathbf{q} = \int_0^\infty \mathbf{q}_\nu d\nu, \quad \mathbf{q}_\nu = \int_{4\pi} \mathbf{\Omega} I_\nu d\Omega \tag{1.2}$$

$$\begin{aligned} &\mathbf{\Omega}\,\mathrm{grad}\, I_\nu (\mathbf{r}, \Omega) = \varkappa_\nu (\mathbf{r}) [B_\nu (T) - I_\nu (\mathbf{r}, \mathbf{\Omega})] \\ &B_\nu = \frac{2h\nu^3}{c^2} (e^{h\nu/kT} - 1)^{-1} \end{aligned} \tag{1.3}$$

where ν is the frequency of radiation, $\mathbf{r}$ is the radius vector of a point, $\mathbf{\Omega}$ is a unit vector in the direction of a light ray, $\varkappa_\nu$ is the volume absorption coefficient, and B_ν is the Planck function.

The equations of state and the dependence of $\varkappa$ on ρ and s

$$p = p (\rho, s), \quad T = T (\rho, s), \quad \varkappa_\nu = \varkappa_\nu (\rho, s) \tag{1.4}$$

close the system of Eqs. (1.1) - (1.4).

Equations (1.1) with allowance for the equation of state (1.4) yield the corollaries [2]

$$\begin{aligned} &(a_s^2 - u^2) \frac{\partial u}{\partial x} - uv \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + (a_s^2 - v^2) \frac{\partial v}{\partial y} + \\ &\quad \frac{d-1}{y} v a_s^2 = -\frac{1}{\rho^2 T} \left(\frac{\partial p}{\partial s} \right)_\rho \mathrm{div}\, \mathbf{q} \\ &(a_T^2 - u^2) \frac{\partial u}{\partial x} - uv \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + (a_T^2 - v^2) \frac{\partial v}{\partial y} + \frac{d-1}{y} v a_T^2 = \\ &\quad \frac{1}{\rho} \left(\frac{\partial p}{\partial T} \right)_\rho \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) \end{aligned} \tag{1.5}$$

where a_s and a_T denote the isentropic and isothermal speeds of sound which satisfy the relationships [2]

$$a_s^2 = a_T^2 + \frac{c_V}{\rho^2 T} e_{12}^2, \quad e_{12} = -\sqrt{\frac{\gamma - 1}{\gamma} \frac{\rho^2 a_s^2 T}{c_V}}, \quad \gamma = \frac{c_p}{c_V}$$

$$a_s = (\partial p / \partial \rho)_s^{1/2}, \quad a_T = (\partial p / \partial \rho)_T^{1/2}$$

where c_p and c_V are the specific heats of gas at constant pressure and volume, respectively.

Henceforth Eqs. (1.5) will be used in system (1.1) - (1.4) instead of the first and fourth of Eqs. (1.1). The new system of equations is equivalent to the input one.

2. Derivation of approximate equations. Let the profile f and temperature T_w be specified by formulas

$$y = L_y + \delta L_x f (x / L_x), \quad T_w = T_0 (1 + \varepsilon_w T_w' (x / L_x))$$

where L_x is a characteristic size of the perturbation of wall temperature or shape, L_y is the channel width at inlet, T_0 is the temperature of the oncoming stream and δ and ε_w are small parameters.

Let functions f and T_w' be of order $O(1)$ for finite values of the argument and let these satisfy at infinity the limiting relationships

$$f(-\infty) = T_w'(-\infty) = 0, \quad df(\infty)/dx = dT_w(\infty)/dx = 0$$

We shall consider motions of a gas whose speeds a_s and a_T are close to each other. Unperturbed quantities will be denoted by subscripts zero and the perturbed ones by a prime.

We represent the thermodynamic derivative e_{120} as [2]

$$e_{120} = \varepsilon_a \frac{r_0}{c_{V_0}} e_{120}', \quad \varepsilon_a \ll 1, \quad e_{120}' \sim 1$$

The velocity of the oncoming stream is assumed to be close to the isentropic and isothermal speed of sound

$$u_0 - a_{s_0} = \varepsilon_a^2 u_0 A_{s_0}, \quad u_0 - a_{T_0} = \varepsilon_a^2 u_0 A_{T_0}$$

Let us assume that the state of gas in the flow regions differs slightly from that of a uniform stream at temperature T_0 and velocity u_0 at inlet. We set $\varepsilon_w = \varepsilon_a \varepsilon_T$ and write down the first terms of the expansion of unknown functions in asymptotic series

$$u = u_0(1 + \varepsilon u'),\ v = \delta u_0 v',\ p = p_0(1 + \varepsilon p'),\ \rho = \rho_0(1 + \varepsilon \rho') \tag{2.1}$$

$$T = T_0(1 + \varepsilon_a \varepsilon_T T'), \quad \mathbf{q} = 16 \varepsilon_a \varepsilon_T \sigma T_0^4 \mathbf{q}' \tag{2.2}$$

$$I_\nu = B_{\nu_0} + \varepsilon_a \varepsilon_T T_0 H_\nu I_\nu', \ \mathbf{q}_\nu = 4\pi \varepsilon_a \varepsilon_T T_0 H_\nu \mathbf{q}_\nu'$$

$$w_\nu = \frac{1}{c} \int_{4\pi} I_\nu \, d\Omega = \frac{4\pi}{c} (B_{\nu_0} + \varepsilon_a \varepsilon_T T_0 H_\nu w_\nu'), \quad H_\nu = \frac{\partial B_\nu}{\partial T_0}$$

where σ is the Stefan-Boltzmann constant and w_ν the spectral density of radiant energy. Substituting expansions (2. 2) into the transfer equation (1. 3), in the first approximation we obtain

$$\operatorname{div} \mathbf{q}' = \varkappa_0 (T' - w')$$

$$w' = \int_0^\infty \varkappa_{\nu_0} H_\nu w_\nu' \, d\nu \Big/ \int_0^\infty \varkappa_{\nu_0} H_\nu \, d\nu, \quad w_\nu' = \frac{1}{4\pi} \int_{4\pi} I_\nu' \, d\Omega$$

$$\varkappa_0 = \int_0^\infty \varkappa_{\nu_0} H_\nu \, d\nu \Big/ \int_0^\infty H_\nu \, d\nu$$

It was shown in [1] that in the case of a slightly inhomogeneous medium the transfer equation can always be approximated by the equation for the effective density of radiant energy w'

$$\Delta w' = 3\varkappa_0^2 (w' - T') \tag{2.3}$$

We introduce the dimensionless coordinates $x^\circ = x / L_x$, and $y^\circ = 3^{1/2} \varkappa_0 y$, and determine the typical optical thicknesses

$$\tau_x = 3^{1/2} \varkappa_0 L_x, \quad \tau_y = 3^{1/2} \varkappa_0 L_y$$

We substitute expansions (2. 1) into the second and third equations of motion (1. 1). For the first terms of expansion we obtain

$$p = -(\rho_0 u_0^2 / p_0) u, \quad p = p(x) \tag{2.4}$$

Here and in what follows we omit the primes which denote deviations from equilibrium and, also, the wave in the symbols of dimensionless coordinates. The first of formulas (2.4) is valid for $\delta\tau_x \ll 1$, while the second holds for $\delta / \tau_x \ll \varepsilon$ and $\delta^2 \ll \varepsilon$. When $\tau_x \sim 1$ these inequalities reduce to $\delta \ll \varepsilon$. When the oncoming stream velocity deviates slightly from that of isothermal speed of sound, we have $\varepsilon_a \varepsilon_T \ll \varepsilon$.

The substitution of expansions (2.1) and (2.2) into Eqs. (1.5) yields

$$\begin{gathered} -2\varepsilon^2 m_0 (u + B_{s_0}) \frac{\partial u}{\partial x} + \delta\tau_x \left(\frac{\partial v}{\partial y} + \frac{d-1}{y} v\right) = \varepsilon_a^2 \varepsilon_T \frac{p_0 e_{120}}{b \rho_0 u_0^2} (T - w) \\ -2\varepsilon^2 m_0 (u + B_{T_0}) \frac{\partial u}{\partial x} + \delta\tau_x \left(\frac{\partial v}{\partial y} + \frac{d-1}{y} v\right) = -\varepsilon_T \varepsilon_a^2 \frac{p_0 e_{120}}{\rho u_0^2} \frac{\partial T}{\partial x} \end{gathered} \tag{2.5}$$

where

$$B_{s_0} = \frac{\varepsilon_a^2}{\varepsilon} \frac{A_{s_0}}{m_0}, \quad B_{T_0} = \frac{\varepsilon_a^2}{\varepsilon} \frac{A_{T_0}}{m_0}, \quad m_0 = \frac{1}{2\rho_0^3 u_0^2} \frac{\partial^2 p}{\partial (1/\rho_0)^2}$$

$$N_{B_0} = \frac{\rho_0 u_0^3}{\sigma T_0^4}, \quad b = \frac{3^{1/2} N_{B_0} c_{V_0} T_0}{\tau_x u_0^2}$$

From the equations of state and the first of formulas (2.4) with $\varepsilon_a \varepsilon_T \ll \varepsilon$ we have

$$\rho + u = 0 \tag{2.6}$$

We subject the small parameters to the following relationships:

$$2\varepsilon^2 m_0 = \delta\tau_x = \varepsilon\varepsilon_a^2 = -\varepsilon_a^2 \varepsilon_T \frac{p_0 e_{120}}{\rho_0 u_0^2} = -2\varepsilon_T \varepsilon \frac{m_0 \rho_0 c_{V_0} T_0}{p_0 e_{120}} \tag{2.7}$$

The last two equalities imply that $B_{T_0} - B_{s_0} = 1$. Taking into account relationships (2.7) and Eqs. (2.4) and (2.5) we obtain

$$-(u + B_{s_0}) \frac{du}{dx} + \frac{\partial v}{\partial y} + (d-1) \frac{v}{y} = \frac{1}{b} (w - T) \tag{2.8}$$

$$-(u + B_{T_0}) \frac{du}{dx} + \frac{\partial v}{\partial y} + (d-1) \frac{v}{y} = \frac{\partial T}{\partial x} \tag{2.9}$$

Subtracting Eq. (2.9) from (2.8) we have

$$\frac{du}{dx} = \frac{1}{b} (w - T) - \frac{\partial T}{\partial x} \tag{2.10}$$

Using dimensionless coordinates we rewrite Eq. (2.3) in the form

$$\frac{1}{\tau_x^2} \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + (d-1) \frac{1}{y} \frac{\partial w}{\partial y} = (w - T) \tag{2.11}$$

Let us consider the boundary conditions for the obtained equations. It was shown in [1] that the boundary conditions for Eq. (2.11) are formulated as follows: on an ideally black wall at temperature T_w $w+(2/3\varkappa_0)\partial w/\partial \mathbf{n} = T_w$ while on an adiabatic wall $\partial w/\partial \mathbf{n} = 0$. In these formulas $\mathbf{n}$ is a normal to the wall surface.

Taking into consideration the shape of the wall we obtain the following conditions for the first approximation: for the ideally black and for the adiabatic wall we have, respectively,

$$y = \tau_y:\ w + (^2/_3\varkappa_0)\partial w/\partial y = T_w \tag{2.12}$$

$$y = \tau_y:\ \partial w/\partial y = 0 \tag{2.13}$$

The wall satisfies the condition of impermeability

$$y = \tau_y:\ v = f'(x) \tag{2.14}$$

Here and in what follows the prime denotes a derivative with respect to the x-coordinate.

Conditions

$$y = 0:\ v = 0, \quad \partial w/\partial y = 0 \tag{2.15}$$

are satisfied at the axis or the plane of symmetry.

When $x = \pm\infty$ the radiation is in equilibrium with the gas and the flow is a uniform stream

$$\begin{aligned} x &= -\infty:\ u = v = 0, \quad w = T = 0 \\ x &= +\infty:\ u = \text{const}, \quad v = 0, \quad w = T = T_w \end{aligned} \tag{2.16}$$

In the first approximation the process is defined by the system of Eqs. (2.8), (2.9) and (2.11) with boundary conditions (2.12) or (2.13) and (2.14)-(2.16). The unknown functions v, T, and w depend on two arguments, while the lengthwise velocity u depends only on the x-coordinate.

Let us consider the inverse problem for the flow of gas in a channel with an ideally black wall. For this we assume that functions $u(x)$ and $f(x)$ are known. For the determination of functions w, v, T, and T_w we then have four equations: (2.8), (2.9), (2.11), and (2.12) with boundary conditions (2.14)-(2.16). The gas pressure p is obtained from formula (2.4) and its density ρ from formula (2.6).

Let us determine the dependence of transverse velocity v on the lengthwise velocity $u(x)$ and on the density of radiant energy w. We multiply (2.8) by b and integrate Eqs. (2.8) and (2.11) with respect to y. We then equate the right-hand sides of obtained equalities taking into account condition (2.15) at the axis of symmetry, and obtain

$$v = \frac{y}{d}(u + B'_{s\cdot})\frac{du}{dx} + \frac{1}{b}\left[\frac{\partial w}{\partial y} + \frac{1}{\tau_x^2 y^{d-1}}\frac{\partial^2}{\partial x^2}\int_0^y \eta^{d-1} w(x,\eta)\, d\eta\right] \tag{2.17}$$

It follows from the impermeability condition (2.14) that

$$f'(x) = \frac{\tau_y}{d}(u + B_{s0})\frac{du}{dx} + \tag{2.18}$$

$$\frac{1}{b}\left[\frac{\partial w(x,\tau_y)}{\partial y} + \frac{1}{\tau_x^2 \tau_y^{d-1}}\frac{\partial^2}{\partial x^2}\int_0^{\tau_y} \eta^{d-1} w(x,\eta)\, d\eta\right]$$

Parameter b defines the effect of radiation on the motion of gas. When $b = \infty$ there is no radiation, and from formulas (2.17), (2.18), and (2.10) we have

$$v = (y / d)(u + B_{s0})du / dx, \quad f = (\tau_y / 2d)(u + 2B_{s0}) u, \quad u = -T \quad (2.19)$$

In this case the gas temperature depends only on the x-coordinate, and the transverse velocity v is a linear function of y. Formulas (2.19) define the flow of gas in a hydraulic approximation [3].

3. The case of considerable τ_x. Let us consider the case when the optical thickness $\tau_x \gg 1$ The transfer equation (2.11) is then of the form

$$\frac{\partial^2 w}{\partial y^2} + \frac{d-1}{y}\frac{\partial w}{\partial y} = w - T \quad (3.1)$$

Equation (3.1) defines one-dimensional radiation in the transverse direction. Substitution of the expression for temperature T in Eq. (3.1) into (2.10) yields

$$\frac{\partial w}{\partial x} = \left(\frac{\partial^2}{\partial y^2} + \frac{d-1}{y}\frac{d}{dx}\right)\left(\frac{\partial w}{\partial x} + \frac{w}{b}\right) - \frac{du}{dx} \quad (3.2)$$

This equation, which links the density of radiant energy w with the lengthwise velocity u, was considered in the one-dimensional linear theory of nonstationary radiation transfer in [4, 5] for $d = 1$. The x-coordinate represents there time and the y coordinate the optical thickness.

When $\tau_x \gg 1$ formulas (2.17) and (2.18) are also simplified

$$v = (y / d)(u - u_s)u' + (1 / b)\partial w / \partial y \quad (3.3)$$

$$f'(x) = (\tau_y / d)(u - u_s)u' + (1 / b)\partial w (x, \tau_y) / \partial y \quad (3.4)$$

where $u_s = -B_{s0}$ and $u_T = -B_{T0}$ denote gas velocities at points where conditions $u(x) = a_s(x)$ and $u(x) = a_T(x)$, respectively, are satisfied. Note that $u_s - u_T = 1$.

We introduce notation

$$\varphi_s (u, f) = f - {}^1/_2 (\tau_y / d)\; u\,(u - 2u_s) \quad (3.5)$$

$$\varphi_T (u, f) = f - {}^1/_2 (\tau_y / d)\; u\,(u - 2u_T)$$

Using notation (3.5) it becomes possible to express the impermeability condition (3.4) in the form

$$y = \tau_y: \quad \partial w / \partial y = b\varphi_s'$$

For the density of radiant energy w we have the nonhomogeneous equation (3.2) with boundary conditions

$$y = \tau_y: \partial w / \partial y = b\varphi_s'; \; y = 0: \partial w / \partial y = 0; \; x = -\infty: w = 0 \quad (3.6)$$

We solve here the inverse problem in which u and f are specified functions of argument x. The substitution $W = w - (b / \tau_y)(d + y^2 / 2)\varphi_s' - (d / \tau_y)\varphi_T$ yields homogeneous boundary conditions and a nonhomogeneous equation for function W

$$y = \tau_y: \partial W / \partial y = 0; \quad y = 0: \partial W / \partial y = 0; \quad x = -\infty: W = 0 \quad (3.7)$$

$$\frac{\partial W}{\partial x}=\left(\frac{\partial^2}{\partial y^2}+\frac{d-1}{y}\frac{\partial}{\partial y}\right)\left(\frac{\partial W}{\partial x}+\frac{W}{b}\right)-\frac{b}{2\tau_y}y^2\varphi_s'' \tag{3.8}$$

The first condition (3.6) is obtained from the impermeability condition (3.4), not from the energy balance (2.12) at the wall.

Let us consider the plane case when $d = 1$. We seek the solution of problem (3.7), (3.8) in the form of Fourier series.

$$W=\frac{1}{2}W_0+\sum_{n=1}^{\infty}W_n(x)\cos\alpha_n y,\quad \alpha_n=\frac{\pi n}{\tau_y} \tag{3.9}$$

The coefficients of $W_n(x)$ satisfy the differential equation

$$(1+\alpha_n^2)W_n'+(1/b)\alpha_n^2W_n+{}^1/_2\,(b/\tau_y)\,c_n\varphi_s''=0,\quad n=0,1,2\ldots \tag{3.10}$$

$$c_0={}^2/_3\tau_y^2,\quad c_n=(-1)^n\cdot 4/\alpha_n^2,\quad n\geqslant 1$$

where c_n are the Fourier coefficients of function y^2. Boundary conditions (3.7) and series (3.9) are automatically satisfied. Equations (3.10) have the following solutions:

$$W_0=-{}^1/_3 b\tau_y\varphi_s'$$

$$W_n(x)=(-1)^{n+1}\frac{2b}{\tau_y\alpha_n^2(1+\alpha_n^2)}\int_{-\infty}^{x}\varphi_s''(\xi)\times$$

$$\exp\left[-\frac{\alpha_n^2}{b(1+\alpha_n^2)}(x-\xi)\right]d\xi,\quad n\geqslant 1.$$

Reverting to function w we obtain

$$w(x,y)=\frac{b}{\tau_y}\left(1+\frac{1}{2}y^2-\frac{1}{6}\tau_y^2\right)\varphi_s'+\frac{1}{\tau_y}\varphi_T+\sum_{n=1}^{\infty}W_n(x)\cos\alpha_n y \tag{3.11}$$

The energy balance (2.12) and formulas (3.11) yield the wall temperature

$$T_w(x)=\frac{b_0}{\tau_y}\varphi_s'+\frac{1}{\tau_y}\varphi_T-b\tau_y^3\int_{-\infty}^{x}\varphi_s'''(\xi)\,Q(b,\tau_y;x-\xi)\,d\xi \tag{3.12}$$

$$b_0=b\,(1+(2/\sqrt{3})\tau_y+{}^1/_3\tau_y^2)$$

$$Q(b,\tau_y;\xi)=\frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2(\tau_y^2+n^2\pi^2)}\exp\left[-\frac{n^2\pi^2}{b(\tau_y^2+n^2\pi^2)}\xi\right]$$

Let us estimate the series appearing in (3.11) for the continuous function φ_s'', with the use of the mean value theorem

$$W_n(x)=(-1)^{n+1}\frac{2b}{\tau_y\alpha_n^2(1+\alpha_n^2)}\varphi_s''(x_n)\times$$

$$\int_{-\infty}^{x}\exp\left[-\frac{\alpha_n^2}{b(1+\alpha_n^2)}(x-\xi)\right]d\xi=(-1)^{n+1}\frac{2b^2}{\pi^4n^4}\tau_y^3\varphi_s''(x_n)$$

$$-\infty<x_n<\infty$$

then
$$\left|\sum_{n=1}^{\infty} W_n \cos \alpha_n y\right| \leqslant \sum_{n=1}^{\infty} |W_n(x)\cos\alpha_n y| \leqslant \sum_{n=1}^{\infty} |W_n(x)| =$$
$$\frac{2}{\pi^4} b\tau_y^3 \sum_{n=1}^{\infty} \frac{|\varphi_s''(x_n)|}{n^4} \leqslant \frac{2}{\pi^4} b^2\tau_y^3 m \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{45} mb^2\tau_y^3$$
$$m = \max_{1<n<\infty} |\varphi_s''(x_n)|$$

If the sums in (3. 11) and (3. 12) are small in comparison with the remaining terms, then w and T_w are related to the lengthwise velocity u and function f by formulas
$$w(x, y) = (b / \tau_y)(1 + y^2 / 2 - \tau_y^2 / 6)\varphi_s' + (1 / \tau_y)\varphi_T$$
$$T_w(x) = (b_0 / \tau_y)\varphi_s' + (1 / \tau_y)\varphi_T$$

The sum in (3. 11) is small when the velocity profile is smooth(small m)the radiation effect is small(small b) and the transverse optical thickness of the channel is small(small τ_y)

Let us calculate the gas temperature
$$T = w - \frac{\partial^2 w}{\partial y^2} = \frac{b}{2\tau_y}\left(y^2 - \frac{\tau_y^2}{3}\right)\varphi_s' + \frac{1}{\tau_y}\varphi_T + \sum_{n=1}^{\infty}(1+\alpha_n^2) + W_n(x)\cos\alpha_n y \tag{3.13}$$

and the magnitude of the temperature jump$[T]_w$ at the wall
$$[T]_w = T_w(x) - T(x, \tau_y) =$$
$$\frac{b}{\tau_y}\left(1 + \frac{2}{\sqrt{3}}\tau_y\right)\varphi_s' + 2b\tau_y \int_{-\infty}^{x} \varphi_s''(\xi)\, Q_1(b, \tau_y; x-\xi)\, d\xi$$
$$Q_1(b, \tau_y; \xi) = \sum_{n=1}^{\infty} \frac{1}{\tau_y^2 + \pi^2 n^2} \exp\left[-\frac{n^2\pi^2}{b(\tau_y^2 + \pi^2 n^2)}\xi\right]$$

Solution of the inverse problem is completely determined by formulas(3. 11) and (3. 13).

The inverse problem may be formulated by specifying functions $u(x)$ and $T_w(x)$. To determine $f(x)$ it is necessary to solve the integro-differential equation (3. 12) and, then, determine w and T_w by formulas (3. 11) and (3. 13).

It is seen from (3. 12) that for considerable b the main contribution to T_w.is provided by the first and third terms. Since considerable values of b correspond to a slight effect of radiation on the motion of gas, these terms are associated with isentropic processes. When b is small the main contribution is provided by the term $\varphi_T / \tau_y = -u(u - 2u_T) / 2d + f / \tau_y$, which defines isothermal processes.

Function $T_w(x)$ at $u(x) = 0.5(1 + \operatorname{th} x)$ and $u_s = 0.5$ is shown in Fig. 1 for several values of b. Solid lines relate to a straight wall $f(x) = 0$, and the dash lines to a curved wall with $f(x) = -\exp(-x^2)$.

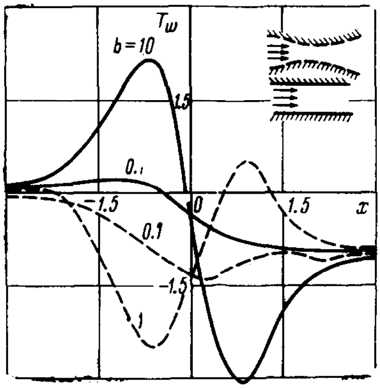

Fig. 1

The distribution of temperature T in the channel for a given acceleration mode is of interest. In that case it is not possible to neglect the series, as was done in the determination of T_w and w. Results of calculations are presented in Figs. 2 and 3, where solid lines correpond to function $f = 0$ and the dash lines to $f = -\exp(-x^2)$.

Figure 2 corresponds to the same acceleration mode as Fig. 1, viz. $u(x) = 0.5\,(1 + \operatorname{th} x)$ and $u_s = 0.5$, while Fig. 3 relates to velocity $u(x) = 1.5\,(1 + \operatorname{th} x)$ and $u_s = 2.5$.

In the first case the gas velocity passes only through isentropic speed of sound u_s and in the second case it passes through u_s and the isothermal speed of sound u_T. The wall temperature T_w in Fig. 3 is considerably higher than in Fig. 2.

Relationship between the wall temperature T_w and velocity $u(x)$ is shown in Fig. 4, where curves of function T_w are plotted for the following six different modes with $b = 1$ and $f = 0$.

1) Acceleration from isentropically supersonic to isentropically supersonic velocity $u = 0.5\,(1 + \operatorname{th} x)$, $u_s = -0.5$, and $u_T = -1.5$;

2) acceleration from isentropically subsonic but isothermally supersonic to isentropically supersonic velocity $u = 0.5\,(1 + \operatorname{th} x)$, $u_s = -0.5$, and $u_T = -0.5$;

3) acceleration from isothermally supersonic but isentropically subsonic to isentropically subsonic but isothermally supersonic velocity $u = 0.2\,(1 + \operatorname{th} x)$, $u_s = 0.5$, and $u_T = -0.5$;

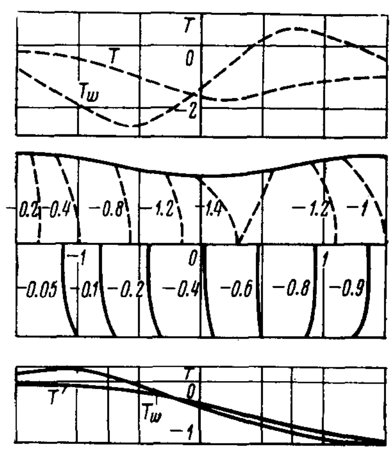

Fig. 2

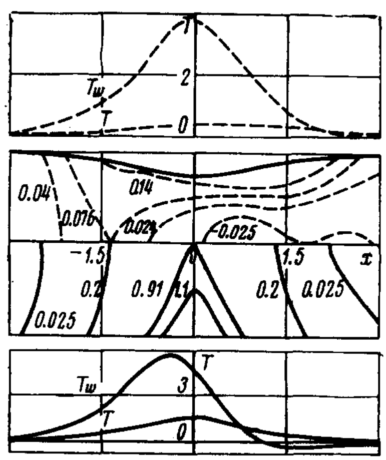

Fig. 3

4) acceleration from isothermally subsonic to isentropically supersonic velocity $u = 1.5\,(1 + \mathrm{th}\, x)$, $u_s = 2.5$, and $u_T = 1.5$.

5) acceleration from isothermally subsonic to isothermally supersonic but isentropically subsonic velocity $u = 0.5\,(1 + \mathrm{th}\, x)$, $u_s = 1.5$, and $u_T = 0.5$; and

6) acceleration from isothermally subsonic to isothermally subsonic velocity $u = 0.5\,(1 + \mathrm{th}\, x)$, $u_s = 2.5$, and $u_T = 1.5$;

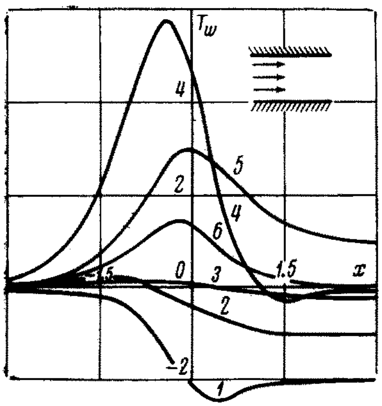

Fig. 4

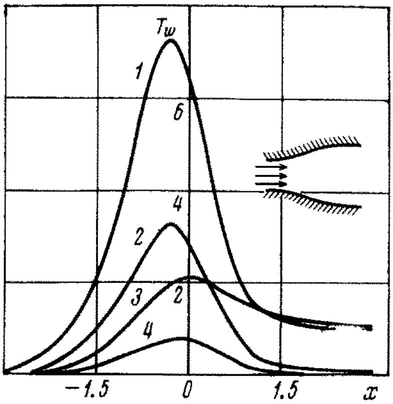

Fig. 5

In Fig. 5 is shown the acceleration of gas by radiation when the wall geometry hinders the acceleration $f = 0.5\,(1 + \mathrm{th}\, x)$. Curves 1 and 3 relate to velocity $u\,(x) = 1.5\,(1 + \mathrm{th}\, x)$, $u_s = 2.5$, while curves 2 and 4 correspond to velocity $u\,(x) = (1 + \mathrm{th}\, x)$, $u_s = 1.5$. Curves 1 and 2 are for parameter $b = 1$, and curves 3 and 4 are for $b = 0.1$

Let us consider the gas flow in a channel with an adiabatic wall. In this case temperature T_w does not appear in the problem. It follows from condition (2.13) and the stipulation of impermeability that f and u are linked by the relationship $f = (\tau_y \,/\, 2d)\, u\, (u + 2B_{s0})$ Equation (3.2) with the homogeneous condition (3.6) has the solution $w = -u$. It follows from (3.1) that $T = -u$, and from (3.3) that $v = (y \,/\, d)(u - u_s)u'$. The derived solution coincides with solution (2.9) for the flow of nonradiative gas. Hence in the first approximation an adiabatic wall excludes the effect of radiation on the flow of gas.

If the profile f and velocity u are specified so that equality $\varphi_s = f - (\tau_y \,/\, 2d)(u - 2u_s)u = 0$, is satisfied, a solution coincident with the hydraulic approximation is again obtained.

4. The case of $\tau_x \sim 1$. Let us consider the inverse problem for the flow of gas in a channel with an ideally black wall for $\tau_x \sim 1$. Using the procedure applied in Sect. 3 we obtain the expression for T and the equation for w

$$T = w - \frac{1}{\tau_x^2}\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} - \frac{d-1}{y}\frac{\partial w}{\partial y} \tag{4.1}$$

$$\frac{\partial w}{\partial x} - \frac{1}{\tau_x^2}\frac{\partial^2}{\partial x^2}\left(\frac{\partial w}{\partial x} + \frac{w}{b}\right) = \left(\frac{\partial^2}{\partial y^2} + \frac{d-1}{y}\frac{\partial}{\partial y}\right)\left(\frac{\partial w}{\partial x} + \frac{w}{b}\right) - \frac{du}{dx}$$

Integrating Eqs. (2.8), (2.9) and (2.11) from 0 to τ_y and using the impermeability condition at $y = \tau_y$, we obtain

$$\partial w / \partial y = \psi = b\varphi_s' - (1 / \tau_x^2)(b\varphi_s' + \varphi_T)''$$

The substitution $w = (y^2 / 2\tau_y)\psi + w_0$ yields an equation with homogeneous boundary conditions

$$\frac{\partial w_0}{\partial x} - \frac{1}{\tau_x^2}\frac{\partial^2}{\partial x^2}\left(\frac{\partial w_0}{\partial x} + \frac{w_0}{b}\right) = \left(\frac{\partial^2}{\partial y^2} + \frac{d-1}{y}\frac{\partial}{\partial y}\right)\left(\frac{\partial w_0}{\partial x} + \frac{w_0}{b}\right) \tag{4.2}$$

$$\frac{y^2}{2\tau_y}\left[\psi - \frac{1}{\tau_y^2}\left(\psi' + \frac{\psi}{b}\right)'\right]' + \frac{d}{\tau_y}\left(\psi' + \frac{\psi}{b}\right) - u'$$

$$y = 0: \frac{\partial w_0}{\partial y} = 0, \quad y = \tau_y: \frac{\partial w_0}{\partial y} = 0$$

We seek the solution of Eq. (4.2) of the form

$$w_0 = w_1(x, y) + w_2(x)$$

For $w_2(x)$ from (4.2) we obtain

$$w_2' - \frac{1}{\tau_x^2}\left(w_2' + \frac{w_2}{b}\right)'' = -u' + \frac{d}{\tau_y}\left(\psi' + \frac{\psi}{b}\right) \tag{4.3}$$

Substituting the expressions for ψ and ψ' into (4.3) we obtain

$$w_2 = (d / \tau_y)(b\varphi_s' + \varphi_T) \tag{4.4}$$

Function $w_1(x, y)$ satisfies the nonhomogeneous equation with homogeneous boundary conditions

$$\frac{\partial w_1}{\partial x} - \frac{1}{\tau_x^2}\frac{\partial^2}{\partial x^2}\left(\frac{\partial w_1}{\partial x} + \frac{w_1}{b}\right) = \left(\frac{\partial^2}{\partial y^2} + \frac{d-1}{y}\frac{\partial}{\partial y}\right)\left(\frac{\partial w_1}{\partial x} + \frac{w_1}{b}\right) \tag{4.5}$$

$$\frac{y^2}{2\tau_y}\left[\psi - \frac{1}{\tau_x^2}\left(\psi' + \frac{\psi}{b}\right)'\right]'$$

$$y = 0: \ \partial w_1 / \partial y = 0, \quad y = \tau_y: \ \partial w_1 / \partial y = 0$$

We set $d = 1$ and seek the solution of Eq. (4.5) in the form of Fourier series

$$w_1(x, y) = \frac{1}{2} w_{10} + \sum_{n=1}^{\infty} w_{1n}(x) \cos \alpha_n y$$

For w_{1n} we obtain the equation

$$\begin{aligned} &w_{1n}''' + (1 / b)w_{1n}'' - \tau_x^2 (1 + \alpha_n^2)w_{1n}' - (\alpha_n^2\tau_x^2 / b)w_{1n} \\ &= (c_n\tau_x^2 / 2\tau_y)[\psi - (1 / \tau_x^2)(\psi' + \psi / b)']' \end{aligned} \tag{4.6}$$

from which at $n = 0$ have $w_{10} = -{}^1/_3\tau_y\psi$. When $n \geqslant 1$ we use the Fourier transformation for solving Eq. (4.6). We have

$$w_{1n}(x) = (-1)^n \frac{2}{\pi^2 n^2} \tau_x^2\tau_y \int_{-\infty}^{+\infty} f(\xi)\, G(x - \xi)\, d\xi$$

$$f = [\psi - (1 / \tau_x^2)(\psi' + \psi / b)']', \quad G_T(\xi) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{ik\xi}}{P_3(ik)}$$

$$P_3(\xi) = \xi^3 + \xi^2 / b - \tau_x^2 (1 + \alpha_n^2)\xi - (\alpha_n^2\tau_x^2 / b)$$

The properties of roots $\gamma_1, \gamma_2, \gamma_3$ of the cubic equation $P_3(\gamma) = 0$.

$$\gamma_1 + \gamma_2 + \gamma_3 = -\frac{1}{b}, \quad \frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \frac{1}{\gamma_3} = -\frac{b(1+\alpha_n^2)}{\alpha_n^2},$$

$$\gamma_1\gamma_2\gamma_3 = \frac{\alpha_n^2\tau_x^2}{b}$$

imply that the real root γ_1 is positive, while the complex-conjugate roots γ_2 and γ_3 have negative real parts. Hence [6]

$$G(\xi) = \begin{cases} -\left[\dfrac{\exp(\gamma_2\xi)}{(\gamma_2-\gamma_1)(\gamma_2-\gamma_3)} + \dfrac{\exp(\gamma_3\xi)}{(\gamma_3-\gamma_1)(\gamma_3-\gamma_2)}\right], & \xi > 0 \\ -\dfrac{\exp(\gamma_1\xi)}{(\gamma_1-\gamma_2)(\gamma_1-\gamma_3)} & \xi < 0 \end{cases}$$

Reverting to function w we obtain

$$w = \left(\frac{b}{\tau_y}\right)\left(1 + \frac{y^2}{2} - \frac{\tau_y^2}{6}\right)\varphi_s'' + \frac{\varphi_T}{\tau_y} - \left(\frac{1}{2}\tau_x^2\tau_y\right) \times \tag{4.7}$$
$$\left(y^2 - \frac{\tau_y^2}{3}\right)(b\varphi_s'' + \varphi_T)'' + \sum_{n=1}^{\infty} w_{1n}(x)\cos\alpha_n y$$

From (2.14) and (2.17) for the wall temperature $[T]_w$ we have

$$T_w = \frac{1}{\tau_y}(b_0\varphi_s' + \varphi_T) - \frac{1}{\tau_x^2}\left(1 + \frac{\tau_y}{3}\right)(b\varphi_s' + \varphi_T)'' + \tag{4.8}$$
$$\tau_x^2\tau_y\left[\int_{-\infty}^{x} f(\xi)\,Q_-(x-\xi)\,d\xi + \int_{x}^{\infty} f(\xi)\,Q_+(\xi - x)\,d\xi\right]$$

$$Q_-(\xi) = \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[\frac{\exp(\gamma_{2n}\xi)}{(\gamma_{1n}-\gamma_{2n})(\gamma_{2n}-\gamma_{3n})} + \frac{\exp(\gamma_{3n}\xi)}{(\gamma_{3n}-\gamma_{1n})(\gamma_{2n}-\gamma_{3n})}\right]$$

$$Q_+(\xi) = \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\frac{\exp(-\gamma_{1n}\xi)}{(\gamma_{1n}-\gamma_{2n})(\gamma_{3n}-\gamma_{1n})}$$

The integral term in (4.8) can be estimated as was done in Sect. 3. We have

$$\tau_x^2\tau_y\left[\int_{-\infty}^{x} f(\xi)\,Q_-(x-\xi)\,d\xi + \int_{x}^{\infty} f(\xi)\,Q_+(\xi-x)\,d\xi\right] \leqslant \frac{1}{45}\tau_y^3 bM$$

$$M = \max_{-\infty<\xi<\infty} f(\xi)$$

The gas temperature T and the temperature jump $[T]_w$ at the wall are obtained, as in Sect. 3, from Eqs. (4.1), (4.7), and (4.8).

Let us consider the flow of gas in a channel with an adiabatic wall. The boundary conditions for w are

$$y = \tau_y:\ \partial w / \partial y = \psi = 0, \quad y = 0:\ \partial w / \partial y = 0$$

Equation (4.5) for $w_1(x, y)$ with homogeneous boundary conditions and zero right-hand side yields a zero solution. Consequently

$$w = w(x) = \frac{d}{\tau_y}\left(b\frac{d\varphi_s}{dx} + \varphi_T\right),$$

$$T = T(x) = \frac{d}{\tau_y}\left(1 - \frac{1}{\tau_x^2}\frac{d^2}{dx^2}\right)\left(b\frac{d\varphi_s}{dx} + \varphi_T\right)$$

$$v = y\frac{d}{dx}\left[f - \varphi_s + \frac{1}{b\tau_y\tau_x^2}\frac{d}{dx}\left(b\frac{d\varphi_s}{dx} + \varphi_T\right)\right]$$

Unlike in Sect. 3 the radiation transfer affects the flow of gas in a channel with adiabatic wall even in the first approximation when $\tau_x \sim 1$. When $\tau_x \to \infty$ all formulas in Sect. 4 coincide with corresponding formulas in Sect. 3.

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ON NONLINEAR DAMPING OF PLANE ACOUSTIC PULSES IN A RADIATIVE GAS

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The nonlinear evolution of small amplitude waves in a viscous heat-conducting gas at low and high Boltzmann radiation number is investigated on the example of the piston problem.